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Tapan Mitra & Fabio Privileggi‡

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Cantor Type Invariant Distributions in the Theory of Optimal Growth under Uncertainty*

TAPAN MITRA[a,†] and FABIO PRIVILEGGI[b,‡]

[a]*Department of Economics, Cornell University, Uris Hall, Ithaca, NY 14853, USA;*
[b]*Department of Public Policy and Public Choice, Via Cavour 84, 15100 Alessandria, Italy*

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We study a one-sector stochastic optimal growth model, where the utility function is iso-elastic and the production function is of the Cobb–Douglas form. Production is affected by a multiplicative shock taking one of two values. We provide sufficient conditions on the parameters of the model under which the invariant distribution of the stochastic process of optimal output levels is of the Cantor type.

Keywords: Stochastic optimal growth, Iterated function system, Invariant measure, No overlap property, Cantor function, Lipschitz policy

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INTRODUCTION

In the theory of optimal economic growth under uncertainty, relatively little is known about the nature of the stochastic steady-state. Mirman and Zilcha [9] considered an example with logarithmic utility function and Cobb–Douglas production function (where a multiplicative random shock to production takes one of two values) to show that the invariant distribution of the stochastic process of outputs would be an absolutely continuous function for some chosen parameter values. Montrucchio and Privileggi [12] considered the same example with different parameter values to show that the invariant distribution of the stochastic process of outputs can be a Cantor function. Mitra *et al.* [10] expanded on this example to establish precise bounds on the parameters of the model under which such Cantor-type and more general *singular* invariant distributions would arise, as well as bounds on the parameters of the model under which the invariant distribution would be *absolutely continuous*.

In the above example, it is well-known that the optimal policy function is linear, and it can be explicitly calculated. This allows one to characterize the nature of the stochastic steady

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†Corresponding author. E-mail: tm19@cornell.edu
‡E-mail: fabio.privileggi@unipmn.it

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state, at least for a wide range of parameter values. However, once one goes beyond this specific example, and allows for instance for the class of iso-elastic utility functions which are *not* of the logarithmic type, the optimal policy function is necessarily non-linear, and its solution in closed form is not known. Thus, the techniques used in the above example to characterize the nature of the invariant distribution are no longer available, and a more general approach is needed.

In this paper, we consider the case of an iso-elastic utility function, and a Cobb–Douglas production function, and we establish suitable sufficient conditions on the parameters of the model under which the invariant distribution is of the Cantor type. While this special framework is maintained throughout, we believe our approach is applicable in more general settings.

In the first part of the paper, we establish a sufficient condition for the crucial "no-overlap property" of the iterated function system (IFS), generated by the optimal policy function, on the stable invariant set of the stochastic process of optimal output. This property leads to an attractor of the IFS resembling a Cantor set.

We develop further properties of the IFS under suitable restrictions on the parameters. Specifically, we provide conditions under which the maps of the IFS are Lipschitz, with Lipschitz constants which can be directly computed, given the parameters of the model. Using the general theory of IFS, we are then able to identify parameter configurations under which the attractor of the IFS has Lebesgue measure zero, so that the invariant distribution is necessarily singular.

We note, in connection with these results, that some of the mathematical literature on IFS (for non-linear maps) has been developed under the condition that the maps of the IFS are twice continuously differentiable on the relevant state space (see, for example, Refs. [7,8]). In our exercise, the IFS is not a primitive, but rather derived from a stochastic dynamic programming problem. For these problems, such smoothness conditions on the resulting maps are in general not possible to establish.[¶] We have, therefore, not used these results, but have instead based our analysis only on those properties of the value and policy functions, which can be established in our framework.

PRELIMINARIES

We consider a special case of the standard model of optimal growth under uncertainty as presented in Refs. [5,9]. Specifically, the production function is one in which the shocks are multiplicative, so there is a function, $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, such that $f(x, r) = rh(x)$ for $(x, r) \in \mathbb{R}_+ \times S$. The set S of values of the random variable, r, is $\{1, q\}$, where $q \in (0, 1)$. We interpret the value 1 of r to be the "normal" state, with q representing a downward production shock, occurring with probability $p \in (0, 1)$. The function, h, is specified to be of the Cobb–Douglas type; that is, there is $\alpha \in (0, 1)$, such that $h(x) = x^{1-\alpha}/(1-\alpha)$ for $x \geq 0$.

[¶]See Refs. [1,13] for results on the C^1 differentiability of the optimal policy function, and the difficulties which arise in establishing C^2 differentiability of the optimal policy function in non-stochastic dynamic programming models. For stochastic dynamic programming models, see Ref. [14] for results on the C^1 differentiability of the optimal policy function. If the random shock has a distribution which is smooth (a condition which is clearly violated in our set-up), then it is possible to show that the optimal policy function is twice continuously differentiable, by using the results of Ref. [4].

The utility function, u, is of the iso-elastic type; that is, there is $\beta \in (0,1)$, such that $u(c) = c^{1-\beta}/(1-\beta)$ for $c \geq 0$. Thus, the "primitives" of our model are the parameters q, p, α, β and δ, the discount factor, each belonging to (0,1).

One can apply the standard theory of stochastic dynamic programming to obtain an (optimal) value function, $V : \mathbb{R}_+ \to \mathbb{R}_+$ and an (optimal) policy function, $g : \mathbb{R}_+ \to \mathbb{R}_+$, which we will interpret as the consumption function. That is, given any output level, $y \geq 0$, the optimal consumption out of this output is given by $g(y)$. The optimal input choice (for production in the next period) is then $[y - g(y)]$. We denote $h[y - g(y)]$ by $G(y)$; it is the output obtained in the next period, when r takes the value 1. Then, $qG(y)$ is the output obtained in the next period, when r takes the value q. We will denote the function qG by H.

Following Refs. [5,9], one can establish several useful properties of the value and policy functions. We summarize these results (without proofs) in the following Proposition.

PROPOSITION 1 *The value function, V, and the policy function, g, satisfy the following properties:*

(i) *V is concave on $\mathbb{R}_+$, and continuous on $\mathbb{R}_{++}$;*
(ii) *g is continuous on $\mathbb{R}_+$ and $0 < g(y) < y$ for $y > 0$;*
(iii) *$g(y)$ and $[y - g(y)]$ are increasing in y on $\mathbb{R}_+$;*
(iv) *V is continuously differentiable on $\mathbb{R}_{++}$, and $V'(y) = u'[g(y)]$ for $y > 0$;*
(v) *for $y > 0$, we have*

$$u'[g(y)] = \delta\{pV'[qG(y)]qh'[y - g(y)] + (1-p)V'[G(y)]h'[y - g(y)]\}; \tag{1}$$

(vi) *for $y > 0$, we have*

$$u'[g(y)] = \delta\{pu'[g(qG(y))]qh'[y - g(y)] + (1-p)u'[g(G(y))]h'[y - g(y)]\}. \tag{2}$$

The optimal policy function leads to the stochastic process:

$$y_{t+1} = r_{t+1}G(y_t) \quad \text{for } t \geq 0. \tag{3}$$

Alternately, one might say that the optimal policy function leads to an IFS $\{H, G; p, 1-p\}$. It is known (from Ref. [5]), that there is a unique invariant distribution, μ, of the Markov process described by Eq. (3), and the distribution of optimal output at date t, call it μ_t, converges weakly to μ.[§] We are principally interested in the nature of this distribution μ. The distribution function corresponding to μ is denoted by F.

It can be checked that the functions G and H have positive fixed points, and all the fixed points are less than $[1/(1-\alpha)]^{(1/\alpha)}$. Denote by a the largest fixed point of H, and by b the smallest fixed point of G. Following Ref. [5], one can establish that $a < b$. The interval $[a, b]$ is an invariant stable set of the stochastic process (3). In particular, the support of F is contained in $[a, b]$. Consequently, in studying the nature of F, it is enough to concentrate on the stochastic process (3), with initial output, $y \in [a, b]$. Equivalently, one need only study the IFS $\{H, G; p, 1-p\}$ on the state space $X = [a, b]$.

[§]For an alternate and simpler approach to this result, see Ref. [2].

THE NO OVERLAP PROPERTY

Let us examine some elementary features of the IFS $\{H, G; p, 1-p\}$ on the state space $X = [a, b]$. First, we look at the function H. We have $H(a) = a$; and, for $y \in (a, b]$, we have $H(y) < y$, so the graph of the map lies below the 45° line (except at a). Further $H(y)$ increases with y, reaching $H(b) < G(b) = b$ at $y = b$. Next, we look at the function G. Clearly, $G(a) = (a/q) > a$; and for all $y \in [a, b)$, we must have $G(y) > y$, so the graph of the map lies above the 45° line (except at b). Further, $G(y)$ increases with y, reaching $G(b) = b$ at $y = b$. The two maps do not overlap if:

$$H(b) < G(a) \tag{4}$$

so that the maximum of the H function is less than the minimum of the G function on the state space $X = [a, b]$.

We want to find conditions on the primitives of the model (q, p, α, β and δ) which ensure the no-overlap property (4).

PROPOSITION 2 *Suppose the following condition is satisfied:*

$$q^{2\alpha-1} < [\delta p(1-\alpha)]^{(1-\alpha)}. \tag{5}$$

Then the IFS $\{H, G; p, 1-p\}$ *on the state space* $X = [a, b]$ *has the no overlap property* (4).

Proof Since $H(b) \equiv qG(b) = qb$ *and* $G(a) = (a/q)$, *the no-overlap condition* (4) *is equivalent to*

$$q^2 < \frac{a}{b}. \tag{6}$$

We thus need to find a lower bound for a and an upper bound for b such that Eq. (6) *holds true.*

Let us write the stochastic Ramsey–Euler equation (2) [*Proposition* 1, (vi)] *at* $y = a$ *and at* $y = b$:

$$\frac{1}{[g(a)]^{\beta}} = \delta\left\{\frac{pq}{[g(a)]^{\beta}\,[a-g(a)]^{\alpha}} + \frac{1-p}{[g(a/q)]^{\beta}\,[a-g(a)]^{\alpha}}\right\} \tag{7}$$

$$\frac{1}{[g(b)]^{\beta}} = \delta\left\{\frac{pq}{[g(qb)]^{\beta}\,[b-g(b)]^{\alpha}} + \frac{1-p}{[g(b)]^{\beta}\,[b-g(b)]^{\alpha}}\right\} \tag{8}$$

Equation (7) *can be re-written as:*

$$\frac{1}{\delta} = \frac{pq}{[a-g(a)]^{\alpha}} + \frac{(1-p)[g(a)]^{\beta}}{[g(a/q)]^{\beta}\,[a-g(a)]^{\alpha}}. \tag{9}$$

Using the fact that $G(a) = (a/q)$, *and so:*

$$[a-g(a)]^{\alpha} = [(1-\alpha)a/q]^{\left(\frac{\alpha}{1-\alpha}\right)} \tag{10}$$

the Eq. (9) *can be further simplified to read:*

$$\frac{1}{\delta}[(1-\alpha)a/q]^{\left(\frac{\alpha}{1-\alpha}\right)} = qp + \frac{(1-p)[g(a)]^{\beta}}{[g(a/q)]^{\beta}}. \tag{11}$$

The right-hand side expression in Eq. (11) *is at least as large as qp. This yields the inequality:*

$$[(1-\alpha)a/q]^{\left(\frac{\alpha}{1-\alpha}\right)} \geq \delta qp. \tag{12}$$

Manipulating the expression in Eq. (12) *leads to the following lower bound on a:*

$$a \geq \frac{q(\delta qp)^{\left(\frac{1-\alpha}{\alpha}\right)}}{(1-\alpha)}. \tag{13}$$

Using the upper bound on b:

$$b \leq [1/(1-\alpha)]^{(1/\alpha)}$$

and Eq. (13)*, we obtain:*

$$q^{(1/\alpha)}[\delta p(1-\alpha)]^{\left(\frac{1-\alpha}{\alpha}\right)} \leq \frac{a}{b}. \tag{14}$$

Then, in view of Eq. (6)*, our sufficient condition for no-overlap is:*

$$q^2 < q^{(1/\alpha)}[\delta p(1-\alpha)]\big(1-\alpha/\alpha\big),$$

which can be rewritten as in Eq. (5)*, thus completing the proof.* □

Remark 1

(i) The sufficient condition (5) is possibly not the sharpest one can obtain. As is clear from the proof, in certain steps we have used somewhat crude bounds. It would be interesting to attempt a complete characterization of the no-overlap property in terms of the primitives of the model.

(ii) A necessary condition for Eq. (5) to hold is that $(2\alpha - 1) > 0$, that is:

$$\alpha > (1/2)$$

which is the condition for no-overlap in the case when the utility function is logarithmic (see Ref. [10]). Since our sufficient condition does not involve the utility coefficient β (and therefore applies to all iso-elastic utility functions) it is to be expected that our condition should turn out to be a stronger restriction than in the logarithmic utility case.

(iii) The sufficient condition (5) is non-vacuous. To see this, note that as $\alpha \to 1$, the right hand side of Eq. (5) converges to 1, while the left hand side of Eq. (5) converges to $q \in (0, 1)$. Thus, for α sufficiently close to 1, condition (5) always holds. That is the no-overlap case arises when the exponent in the Cobb–Douglas production function is "low"; this agrees with the finding in Ref. [10] for the logarithmic utility case.

(iv) For a specific numerical case, choose:

$$q = 0.83, \quad p = 0.9, \quad \delta = 0.9, \quad \alpha = 0.95.$$

Then the right hand side of Eq. (5) can be calculated to be 0.852, while the left hand side of Eq. (5) is 0.846, and condition (5) holds.

THE LIPSCHITZ PROPERTY

We will now show that the IFS $\{H, G; p, 1-p\}$ on $X = [a, b]$ has the Lipschitz property; that is, the maps H and G are Lipschitz continuous on X. It is sufficient for this purpose to show that G is Lipschitz continuous.

The Lipschitz continuity of policy functions in non-stochastic models has been studied by Ref. [11]. Given the structure of our model, we are able to take a fairly direct approach. The Lipschitz constant that we obtain is possibly not the sharpest possible, because (unlike Ref. [11]) we do not incorporate in it an additional term expressing the degree of concavity of the value function. On the other hand, we can therefore bypass the theory linking the concavity of the value function to the concavity of the utility and production functions. This makes our approach simpler, and the Lipschitz constant is seen to directly depend on the exponents of the utility and production functions.

Our purpose in exhibiting the Lipschitz property of the IFS is to obtain a sufficient condition in terms of the primitives of the model (q, p, α, β and δ). It is, therefore, important to obtain a Lipschitz constant which depends only on these parameters and is independent of the points of evaluations of the derivatives of the utility, production and value functions.

Keeping this objective in mind, we first obtain a positive lower bound, expressed in terms of the parameters q, p, α, β and δ, on the optimal propensity to consume, $[g(y)/y]$. This result is clearly also of independent interest as a property of the optimal policy function.

PROPOSITION 3 *Suppose the IFS $\{H, G; p, 1-p\}$ on the state space $X = [a, b]$ has the no overlap property. Further, assume that:*

$$q^2 > \delta[qp + (1-p)](1-\alpha). \tag{15}$$

Then, we have the following lower bound on the optimal propensity to consume:

$$g(y)/y > q^2 - \delta[qp + (1-p)](1-\alpha) \quad \text{for all} \quad y \in X. \tag{16}$$

Proof *Using Eq.* (8)*, and noting that $g(b) > g(qb)$, we have:*

$$\frac{1}{\delta} > \frac{pq}{[b - g(b)]^{\alpha}} + \frac{1-p}{[b - g(b)]^{\alpha}}. \tag{17}$$

On rearranging terms in Eq. (17)*, and denoting $[qp + (1-p)]$ by $\mathbb{E}(r)$ (the expected value of r),*

$$[b - g(b)]^{\alpha} > \delta\mathbb{E}(r). \tag{18}$$

By definition of b, we have:

$$b = G(b) = h[b - g(b)] = \frac{[b - g(b)]^{1-\alpha}}{1-\alpha}.$$

This yields:

$$[b - g(b)]^{\alpha} = [b(1-\alpha)]^{\left(\frac{\alpha}{1-\alpha}\right)}. \tag{19}$$

Combining Eqs. (18) *and* (19)*, we get:*

$$b > \frac{[\delta\mathbb{E}(r)]^{\left(\frac{1-\alpha}{\alpha}\right)}}{1-\alpha}. \tag{20}$$

Using Eq. (9)*, and noting that* $g(a) < g(a/q)$*, we have:*

$$\frac{1}{\delta} < \frac{pq}{[a - g(a)]^{\alpha}} + \frac{(1-p)}{[a - g(a)]^{\alpha}}. \tag{21}$$

On rearranging terms in Eq. (21)*, we get:*

$$[a - g(a)]^{\alpha} < \delta \mathbb{E}[r].$$

This yields the inequality:

$$g(a) > a - [\delta \mathbb{E}(r)]^{1/\alpha}. \tag{22}$$

Thus, for all $y \in X$*, using the fact that g is increasing, we obtain from Eq.* (22)*:*

$$\frac{g(y)}{y} \geq \frac{g(a)}{b} > \frac{a}{b} - \frac{[\delta \mathbb{E}(r)]^{1/\alpha}}{b}. \tag{23}$$

Using Eq. (20)*, the second term on the right hand side expression of Eq.* (23) *is less than* $(1-\alpha)\delta\mathbb{E}(r)$*, while the first term on the right hand side expression of Eq.* (23) *is greater than* q^2*, since the no-overlap property holds. Thus, for all* $y \in X$,

$$g(y)/y > q^2 - (1-\alpha)\,\delta\,\mathbb{E}(r)$$

which establishes the result. □

Remark 2

(i) In particular, if the sufficient condition (5) on the parameters hold, and Eq. (15) holds, then Eq. (16) holds, by Proposition 2. In fact, as is clear from the proof, if Eq. (5) holds, then this itself ensures that Eq. (16) holds. But, this is of interest, of course, only when Eq. (15) holds.

(ii) If we let $\alpha \longrightarrow 1$, given the other parameters of the model (q, p, α, β and δ) fixed, then the sufficient condition (15) is automatically satisfied. In particular, for the numerical example discussed in Remark 1, with $q = 0.83$, $p = 0.9$, $\delta = 0.9$, $\alpha = 0.95$, we have $\delta[qp + (1-p)](1-\alpha) = 0.038115$, and $q^2 = 0.6889$, so Eqs. (5) and (15) are both satisfied. That is, we have the no-overlap property holding, and the propensity to consume has a lower bound of $\{q^2 - \delta[qp + (1-p)](1-\alpha)\} = 0.650785$.

PROPOSITION 4 *Suppose the IFS* $\{H, G; p, 1-p\}$ *on the state space* $X = [a, b]$ *has the no overlap property, and Eq.* (15) *holds. Denote* $\{q^2 - \delta[qp + (1-p)](1-\alpha)\}$ *by m, and define:*

$$L = \frac{\beta(1-m)}{\delta qp[\beta(1-m) + \alpha m]}. \tag{24}$$

Then, for all $y, z \in X$*, we have:*

$$|G(y) - G(z)| \leq L|y - z|. \tag{25}$$

Proof We will first prove that G is locally Lipschitz on X, with Lipschitz constant L. That is, we will show that there is some $\varepsilon > 0$*, such that, whenever* $y, z \in X$*, and* $0 < |y - z| \leq \varepsilon$*, Eq.* (25) *holds with L defined by Eq.* (24)*.*

Denote by m' the minimum value of $[g(y)/y]$ on X; this is well defined by continuity of $[g(y)/y]$ on the (non-empty) compact set X. By Proposition 3, we have $m' > m$. Now, define

$$\lambda = \left[1 - \frac{\varepsilon}{a - g(a)}\right]^{1+\alpha} \tag{26}$$

and choose $\varepsilon > 0$ sufficiently small so that

$$\lambda m' > m. \tag{27}$$

It is sufficient to show that, with this choice of ε, whenever $y, z \in X$ and $0 < z - y \leq \varepsilon$, the inequality (25) *holds.*

So, let us pick arbitrary $y, z \in X$, with $0 < z - y \leq \varepsilon$. Let us write the Eq. (1) [*Proposition* 1 (v)] *at y and at z:*

$$u'[g(y)] = \delta\{pqV'[qG(y)] + (1-p)V'[G(y)]\}h'[y - g(y)] \tag{28}$$

$$u'[g(z)] = \delta\{pqV'[qG(z)] + (1-p)V'[G(z)]\}h'[z - g(z)]. \tag{29}$$

Thus, subtracting Eqs. (29) *from* (28), *we obtain:*

$$u'[g(y)] - u'[g(z)] = \delta\{pqV'[qG(y)] + (1-p)V'[G(y)]\}h'[y - g(y)] \\ - \delta\{[pqV'[qG(z)] + (1-p)V'[G(z)]\}h'[z - g(z)]. \tag{30}$$

Since $G(z) > G(y)$, so that $V'[qG(z)] \leq V'[qG(y)]$ and $V'[G(z)] \leq V'[G(y)]$, we obtain:

$$u'[g(y)] - u'[g(z)] \geq \delta\{pqV'[qG(y)] + (1-p)V'[G(y)]\} \\ \times \{h'[y - g(y)] - h'[z - g(z)]\}. \tag{31}$$

We use the Mean Value theorem to obtain ξ satisfying $g(y) \leq \xi \leq g(z)$, such that:

$$u'[g(y)] - u'[g(z)] = u''(\xi)[g(y) - g(z)]. \tag{32}$$

Similarly, we can find ζ satisfying $[y - g(y)] \leq \zeta \leq [z - g(z)]$, such that:

$$h'[y - g(y)] - h'[z - g(z)] = h''(\zeta)\{[y - g(y)] - [z - g(z)]\}. \tag{33}$$

Using Eqs. (32) *and* (33) *in Eq.* (31) *and changing sign, we obtain:*

$$-u''(\xi)[g(z) - g(y)] \geq -\delta\{pqV'[qG(y)] + (1-p)V'[G(y)]\}h''(\zeta)\{(z - y) - [g(z) - g(y)]\}.$$

Transposing terms, dividing through by $u'[g(y)]$, using Eq. (28)*, and rearranging terms, the last inequality becomes:*

$$\frac{g(z) - g(y)}{z - y} \geq -\frac{h''(\zeta)}{h'[y - g(y)]}\left\{-\frac{u''(\xi)}{u'[g(y)]} - \frac{h''(\zeta)}{h'[y - g(y)]}\right\}^{-1}. \tag{34}$$

It remains to convert the right-hand side into terms involving the parameters of our model.

Note that since ξ satisfies $g(y) \leq \xi$, we have:

$$-u''(\xi) = \frac{\beta}{\xi^{1+\beta}} \leq \frac{\beta}{[g(y)]^{1+\beta}} = -u''[g(y)]$$

so that:

$$-\frac{u''(\xi)}{u'[g(y)]} \le \left\{-\frac{u''[g(y)]g(y)}{u'[g(y)]}\right\}\frac{1}{g(y)} = \frac{\beta}{g(y)}. \tag{35}$$

Since ζ *satisfies* $\zeta \le z - g(z)$, *we have:*

$$-h''(\zeta) = \frac{\alpha}{\zeta^{1+\alpha}} \ge \frac{\alpha}{[z-g(z)]^{1+\alpha}} = -h''[z-g(z)] = -h''[y-g(y)]\left[\frac{y-g(y)}{z-g(z)}\right]^{1+\alpha}.$$

Noting that:

$$\left[\frac{y-g(y)}{z-g(z)}\right]^{1+\alpha} \ge \left[1-\frac{\varepsilon}{a-g(a)}\right]^{1+\alpha} = \lambda$$

we obtain:

$$-\frac{h''(\zeta)}{h'[y-g(y)]} \ge \frac{\alpha\lambda}{y-g(y)}. \tag{36}$$

Using Eqs. (35) *and* (36) *in Eq.* (34) *and rearranging terms, we get:*

$$\frac{g(z)-g(y)}{z-y} \ge \alpha\left[\frac{\lambda g(y)}{y}\right]\left\{\beta\left[1-\frac{g(y)}{y}\right]+\alpha\frac{\lambda g(y)}{y}\right\}^{-1}. \tag{37}$$

Since $[g(y)/y)] \ge m' > m$, *we have* $[\lambda g(y)/y] \ge \lambda m' > m$, *so Eq.* (37) *implies:*

$$\frac{g(z)-g(y)}{z-y} > \frac{\alpha m}{\beta(1-m)+\alpha m}. \tag{38}$$

By definition of G, we have:

$$\begin{aligned}\frac{G(z)-G(y)}{(z-y)} &= \frac{[z-g(z)]^{1-\alpha}-[y-g(y)]^{1-\alpha}}{(1-\alpha)(z-y)}\\ &< \frac{[z-g(z)]-[y-g(y)]}{[y-g(y)]^{\alpha}(z-y)}\\ &< \frac{\beta(1-m)}{[y-g(y)]^{\alpha}[\beta(1-m)+\alpha m]}\end{aligned} \tag{39}$$

where in the first inequality we used the superdifferentiability property of the concave function $f:\mathbb{R}_{++}\to\mathbb{R}_{++}$, *defined by* $f(x)=x^{1-\alpha}$, *and the last inequality follows from Eq.* (38). *Since, by Proposition* 1 (iii), $[x-g(x)]$ *is non-decreasing in x, Eqs.* (10) *and* (12) *imply:*

$$[y-g(y)]^{\alpha} \ge [a-g(a)]^{\alpha} \ge \delta qp$$

and using this information in Eq. (39) *yields:*

$$\frac{G(z)-G(y)}{(z-y)} < \frac{\beta(1-m)}{\delta qp[\beta(1-m)+\alpha m]} = L.$$

This establishes that G is locally Lipschitz on X, with Lipschitz constant L.

It follows from the above result that G is Lipschitz continuous on X with Lipschitz constant L. To see this, pick any $y', z' \in X$, *with* $0 < z' - y'$. *We can find a positive integer N, such that* $N_{\varepsilon} \ge (z'-y')$, *where* ε *was used in the definition of Eq.* (26). *Define* $\eta = (z'-y')/N$;

then $0 < \eta \leq \varepsilon$. *We use* η *to define:*

$$(y_0, y_1, \ldots, y_N) = (y', y' + \eta, \ldots, y' + (N-1)\eta, z').$$

Then, we have, using the fact that G is locally Lipschitz [*with the choice of* ε *used in the definition of Eq.* (26)] *with Lipschitz constant L, and* $0 < (y_{n+1} - y_n) = \eta \leq \varepsilon$ *for* $n = 0, 1, \ldots N-1$,

$$G(z') - G(y') = \sum_{n=0}^{N-1} [G(y_{n+1}) - G(y_n)] \leq L \sum_{n=0}^{N-1} (y_{n+1} - y_n) = L(z' - y').$$

This establishes that G is Lipschitz continuous on X, with Lipschitz constant L. □

Remark 3 Note that the Lipschitz constant L satisfies:

$$L \leq \frac{1-m}{\delta qp(1 - m + \alpha m)} \equiv L'$$

and this gives us a Lipschitz constant L' for G that is independent of the parameter β of the utility function.

CANTOR TYPE INVARIANT DISTRIBUTIONS

Given the results of the previous section, we can apply the standard theory of IFSs to observe that:

(i) there is a unique compact set $A \subset [a, b]$, such that $G(A) \cup H(A) = A$; thus, A is a self-similar set;
(ii) A is the support of the unique invariant distribution, μ, of the Markov process, given by Eq. (3).

Clearly, the set A is of the Cantor type, and the question arises as to whether this Cantor-type set has zero Lebesgue measure. In the following result, we provide a sufficient condition on the parameters of the model under which the set A has zero Lebesgue measure. The sufficient condition used implies in particular that both functions H and G are contractions, but is stronger than this requirement. It seems plausible that the result might be obtained under the weaker sufficient condition that H and G are contractions.

PROPOSITION 5 *Suppose the IFS* $\{H, G; p, 1-p\}$ *satisfies the no-overlap property and the sufficient condition* (15). *Denoting* $\{q^2 - \delta[qp + (1-p)](1-\alpha)\}$ *by m, and* $\beta(1-m)/\delta qp[\alpha m + \beta(1-m)]$ *by L, assume that in addition the following inequality holds:*

$$(1+q)L < 1. \tag{40}$$

Then the support A of the unique invariant distribution μ *of Eq.* (3) *is of Lebesgue measure zero, and* μ *is singular with respect to Lebesgue measure.*

Proof *Define, for the IFS* $\{H, G; p, 1-p\}$,

$$K(H) = \min\{K : K \text{ is a Lipschitz constant of } H\};$$

$$K(G) = \min\{K : K \text{ is a Lipschitz constant of } G\}.$$

The similarity dimension of A is defined to be the (unique) positive root of the equation:

$$[K(H)]^d + [K(G)]^d = 1. \tag{41}$$

Clearly, $K(G) \leq L$, *and* $K(H) \leq qL$. *Given Eq.* (40), *we have:*

$$K(H) + K(G) < 1.$$

Thus, the unique positive root, $\hat{d}$, *of Eq.* (41) *must satisfy* $\hat{d} < 1$. *Thus, the similarity dimension of A is less than* 1. *For a self-similar set, the Hausdorff dimension of the set cannot exceed its similarity dimension (see Ref.* [15], *Theorem* 2.3, *p.* 20*). Thus, the Hausdorff dimension of A is less than* 1.

By definition of Hausdorff dimension, the Hausdorff outer measure of A is zero. Since Lebesgue outer measure coincides with Hausdorff outer measure on $\mathbb{R}$, *the Lebesgue outer measure of A is zero. Since A is closed, it is measurable, and hence the Lebesgue measure of A is zero.*

Since A is the support of μ, *we must have* $\mu(X - A) = 0$, *by definition of support* (*see, for example, Ref.* [6], *p.* 31*). And we have just seen that* $\nu(A) = 0$, *where* ν *is Lebesgue measure. Thus,* μ *is singular with respect to Lebesgue measure (see Ref.* [3], *p.* 374*).* □

Remark 4

(i) The conditions of the Proposition are non-vacuous. If we continue with the numerical example discussed in Remarks 1 and 2, we can check that $L' = 0.5369037$ (where L' is defined in Remark 3), and $(1+q)L' = 0.9825338 < 1$, so that $(1+q)L < 1$. This holds independent of the value of the parameter β of the utility function.

(ii) The formula for L indicates a role of the parameter β of the utility function in generating Cantor type invariant distributions with supports of Lebesgue measure zero. Lower values of β [that is, utility functions with lower elasticity of marginal utility, $\{[-u''(c)]c/u'(c)\}$] would make the Lipschitz constants of the maps H and G lower, leading more readily to singular invariant distributions being generated. This is a new feature that was not possible to ascertain by studying the example with the logarithmic utility function.

CONCLUDING REMARKS

This paper further develops the work started in Ref. [12], and subsequently investigated more thoroughly in Ref. [10] on the nature of the invariant distribution in the standard one-sector optimal growth model under uncertainty.

In the present work, unlike in the previous studies, the optimal policy function cannot be explicitly calculated. So, we develop a more general approach to obtain sufficient conditions on the parameters of the model for the invariant distribution to be a Cantor function.

To obtain such sufficient conditions we rely on the properties of the optimal policy and value functions (Proposition 1). Specifically, a sufficient condition for the no-overlap property of the two maps constituting the IFS (associated with the optimal policy) is given in Proposition 2; this condition is independent of the parameter of the utility function and agrees with the analogous condition, discussed in Ref. [10], to get a Cantor attractor. Moreover, under some additional conditions on the parameters, we are able to establish a lower bound on the optimal propensity to consume (Proposition 3), the Lipschitz property of the IFS (Proposition 4) and the singularity of the invariant distribution with respect to Lebesgue measure (Proposition 5).

The approach used in this paper is potentially applicable to the more general setting of Ref. [5], where (unlike in the present study) the utility and production functions are not necessarily iso-elastic. By improving the estimate of the largest fixed point of the lower map and the smallest fixed point of the upper map of the IFS, and by establishing a Lipschitz property for the IFS, some general conditions could be obtained under which the invariant distribution of the model is singular. We hope to report on this line of research in the future.

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